The lagrangian description of representations of the Poincare group

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The construction of lagrangians describing the various representations of the Poincare group is given in terms of the BRST approach.

1. THE DESCRIPTION OF CONSTRAINTS

It is well known that the particles with the value of spin more than two arise naturally when quantizing such classical objects as the relativistic oscillator [1], string [2] or discrete string [3,4]. The challenging problem for these kinds of theories is to construct the lagrangian description both for free and for interacting particles with the higher spins.

In order to kill ghosts and all other superfluous representations the field theoretical lagrangians, describing irreducible Poincare representations must possess some gauge invariance. Along with the basic fields such lagrangians in general include additional ones. The role of these fields is to single out the irreducible representation of the Poincare group. Some of them are auxiliary, others can be gauged away. After a gauge fixing and solving the equations of motion for auxiliary fields one is left with the only es-

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sential field, describing the irreducible representation of the Poincare group. In general this field corresponds to the Young table with k rows and is described by $\Phi^{(k)}_{\mu_1\mu_2\cdots\mu_{n_1},\nu_1\nu_2\cdots\nu_{n_2},\cdots,\rho_1\rho_2\cdots\rho_{n_k}}(x)$ which is the $n_1 + n_2 + \cdots + n_k$ rank tensor field symmetrical with respect to the permutations of each type of indices. In addition, this field is subject to the mass shell condition and transversality conditions for each type of indices. Further, all traces of the basic field must vanish. The correspondence with a given Young table implies, that after symmetrization of all indices from i - th row with one index from j - th row $(i < j)$ the basic field vanishes, for example

$$\Phi^{(k)}_{\{\mu_1\mu_2\cdots\mu_{n_1},\nu_1\}\nu_2\cdots\nu_{n_2},\cdots,\rho_1\rho_2\cdots\rho_{n_k}}(x) = 0. \qquad (1)$$

To describe all irreducible representations of the Poincare group simultaneously it is convenient to introduce an auxiliary Fock space generated by the creation and annihilation operators a^{i+}_μ, a^j_μ with Lorentz index $\mu = 0, 1, 2, ..., D - 1$ and additional internal index $i = 1, 2, ..., k$. These operators satisfy the following commutation relations

$$\left[a^i_\mu, a^{j+}_\nu\right] = -g_{\mu\nu}\delta^{ij}, \qquad (2)$$

$$g_{\mu\nu} = diag(1, -1, -1, ..., -1), \qquad (3)$$

where δ^{ij} is usual Cronecker symbol.

The general state of the Fock space depends on the space - time coordinates x_μ

$$|\Phi\rangle = \sum \Phi^{(k)}_{\mu_1\mu_2\cdots\mu_{n_1},\cdots,\rho_1\rho_2\cdots\rho_{n_k}}(x) \times$$
$$a^{1+}_{\mu_1} a^{1+}_{\mu_2} \cdots a^{1+}_{\mu_{n_1}} \cdots a^{k+}_{\rho_1} a^{k+}_{\rho_2} \cdots a^{k+}_{\rho_{n_k}} |0\rangle \qquad (4)$$

and the components $\Phi^{(k)}_{\mu_1\mu_2\cdots\mu_{n_1},\cdots,\rho_1\rho_2\cdots\rho_{n_k}}(x)$ are automatically symmetrical under the permutations of indices of the same type. The norm of states in this Fock space is not positively definite due to the minus sign in the commutation relation (2) for the time components of the creation and annihilation operators. The transversality conditions for the components leading to the positively definite norm of physical states are equivalent to the following constraints on the physical vectors of the Fock space

$$L^i|\Phi\rangle = 0, \quad L^i = a^i_\mu p_\mu. \qquad (5)$$

These operators L^i along with their conjugates $L^{i+} = a^{i+}_\mu p_\mu$ and mass shell operator p^2_μ form the following algebra with only nonvanishing commutator

$$\left[L^i, L^{j+}\right] = -p^2_\mu \delta^{ij}. \qquad (6)$$

This simple algebra was considered in [5] in the framework of the BRST approach, which automatically leads to the appearance of all auxiliary fields in the lagrangian. Since the constraints are of the first class the corresponding nilpotent BRST charge can be constructed straightforwardly. As a result the description of mixed symmetry (not only totally symmetric) fields was obtained. However, all these fields describe the reducible representations of the Poincare group due to the absence of tracelessness and mixed symmetry conditions in the initial system of the constraints.

On the other hand, the same algebra of operators arises in the case of massive particles. The only difference is that the right hand side of the relation (6) is now nonvanishing operator. Moreover, this operator can have different eigenvalues $p^2_\mu = m^2_n$ for the different physical states. Therefore the procedure of construction of the nilpotent BRST charge for this simple system of constraints

is complicated , since now constraints $L^{i\pm}$ are of the second class.

The tracelessness conditions correspond in the Fock space to the constraints

$$L^{ij}|\Phi\rangle = 0, \qquad (7)$$

with $L^{ij} = a^i_\mu a^j_\mu$, for $i \neq j$ and $L^{ij} = \frac{1}{2}a^i_\mu a^j_\mu$, for $i = j$, while the mixed symmetry properties follow from the constraints

$$T^{ij}|\Phi\rangle = 0, \qquad i < j, \qquad (8)$$

with $T^{ij} = a^{i+}_\mu a^j_\mu$. The operators L^{ij}, L^{ij+} $(i,j$ are arbitrary) and T^{ij}, T^{ij+}, $(i \neq j)$, along with the additional operators

$$H^i = -T^{ii} + \frac{D}{2} = -a^{i+}_\mu a^i_\mu + \frac{D}{2}, \qquad (9)$$

form the Lie algebra $SO(k+1, k)$. The rank of this algebra is k and corresponding Cartan subalgebra contains all operators H^i. One can choose the operators L^{11} and $T^{i,i+1}$ as k simple roots. The positive and negative roots are, correspondingly, L^{ij}, T^{rs}, $(1 \leq r < s \leq k)$ and L^{ij+}, T^{rs}, $(1 \leq s < r \leq k)$. It means, that the tracelessness and the mixed symmetry conditions are equivalent to annihilation of physical states in the total Fock space by the positive roots of the Lie algebra $SO(k+1, k)$. As it can be easily seen, the Cartan generators (9) are strictly positive in the Fock space and have to be excluded from the total set of constraints. Therefore the standard BRST charge has to be modified for the given realization of the $SO(k+1, k)$ algebra.

In the present talk we give the lagrangian description of the massive reducible representations of the Poincare group and its irreducible massless representations with the corresponding Young tableaux having two or one rows. The later lagrangian can be straightforwardly reduced to the lagrangian of [6], describing the irreducible massless higher spin fields (see also [7]). Though in both cases the corresponding system of constraints contains both the ones of the first and the second class, the nontrivial structure of trilinear ghost terms in the BRST charge makes it nilpotent. The method used for these constructions can be generalized for arbitrary representations of Poincare group as well.

2. REDUCIBLE MASSIVE CASE

To demonstrate the general method of BRST construction with the second class constraints of the considered type we begin with the simplest example of nonvanishing mass. In this section we consider the system with constraints

$$L^0 = -p_\mu{}^2 + m^2, \tag{10}$$

$$L^1 = p_\mu a_\mu, \quad L^{1+} = p_\mu a_\mu^+, \tag{11}$$

This system is intermediate between the systems in [5] and [8] because it describes the massive particle together with all its daughter (due to the absence of the constraints $L^{\pm 11}$ in the theory are present particles with lower spins).

The commutation relation (6) (we are considering the case of one oscillator i.e., $i, j = 1$) means that $L^{\pm 1}$ are the second class constraints. The simplest way to construct BRST - charge with corresponding constraints [9] – [10] is to consider massless case in $D+1$ dimensions with constraints

$$L^0 = -p_\mu^2 + p_D^2, \quad \mu = 0, 1, ..., D-1, \tag{12}$$

$$L^1 = p_\mu a_\mu - p_D a_D, \tag{13}$$

$$L^{1+} = p_\mu a_\mu^+ - p_D a_D^+. \tag{14}$$

Following to the standard procedure we introduce additional set of anticommuting variables η_0, η_1, η_1^+ having ghost number one and corresponding momenta $\mathcal{P}_0, \mathcal{P}_1^+, \mathcal{P}_1$ with commutation relations:

$$\{\eta_0, \mathcal{P}_0\} = \{\eta_1, \mathcal{P}_1^+\} = \{\eta_1^+, \mathcal{P}_1\} = 1. \tag{15}$$

and consider the total Fock space generated by creation operators $a_\mu^+, a_D^+, \eta_1^+, \eta_0, \mathcal{P}_1^+$. In terms of the nilpotent BRST - charge which corresponds to the given system of constraints

$$Q = \eta_0 L^0 + \eta_1^+ L^1 + \eta_1 L^{1+} - \eta_1^+ \eta_1 \mathcal{P}_0 \tag{16}$$

the BRST invariant lagrangian can be written as follows

$$L = -\int d\eta_0 \langle \chi | Q | \chi \rangle. \tag{17}$$

In order to describe the massive particle we fix the following x_D dependence of the Fock space vector $|\chi\rangle$ in (17):

$$|\chi\rangle = U|\chi'\rangle = e^{ix_D m}|\chi'\rangle. \tag{18}$$

The result of the substitution of (18) in the expression (17) is

$$L = -\int d\eta_0 \langle \chi' | \tilde{Q} | \chi' \rangle, \tag{19}$$

The new BRST - charge $\tilde{Q}$ is nilpotent due to unitarity of the transformation $\tilde{Q} = U^{-1}QU$. Our choice of x_D- dependence in the exponent in (18) leads to the presence in the BRST charge of the correct massive operator $\tilde{L}_0 = -p_\mu^2 + m^2$ The same trick with the dimensional reduction will be applied in the consequent sections to more complicated cases. It is exactly the heart of the approach, which makes it possible to construct the nilpotent BRST charge in the presence of the second class constraints. The nilpotency of the BRST - charge evidently does not depend from the unitary transformation of the type (18).

In order to be physical, the lagrangian must have zero ghost number. It means that the most general expressions for the vector $|\chi'\rangle$ is

$$|\chi'\rangle = |S_1\rangle + \eta_1^+ \mathcal{P}_1^+ |S_2\rangle, +\eta_0 \mathcal{P}_1^+ |S_3\rangle, \tag{20}$$

with vectors $|S_i\rangle$ having ghost number zero and depending only on bosonic creation operators a_μ^+, a_D^+

$$|S_i\rangle = \sum \phi_{\mu_1,\mu_2,...\mu_n}^n(x) a_{\mu_1}^+ a_{\mu_2}^+ ... a_{\mu_n}^+ (a_D^+)^n |0\rangle. \tag{21}$$

The nilpotency of the BRST - charge leads to the invariance of the lagrangian (19) under the following transformations

$$\delta|\chi'\rangle = \tilde{Q}|\Lambda\rangle. \tag{22}$$

The parameter of transformation must have ghost number -1 and can be written as $|\Lambda\rangle = \mathcal{P}_1^+|\lambda\rangle$, where $|\lambda\rangle$ belong to the Fock space generated by a_μ^+, a_D^+ and depends on the space - time coordinates.

After the substitution of equation (20) into the lagrangian (19) and integration with respect to η_0 the lagrangian is expressed only via the fields $|S_i\rangle$

$$\begin{aligned} L = &-\langle S_1|\tilde{L}^0|S_1\rangle + \langle S_2|\tilde{L}^0|S_2\rangle + \\ &\langle S_1|\tilde{L}^{1+}|S_3\rangle + \langle S_3|\tilde{L}^1|S_1\rangle - \langle S_3||S_3\rangle - \\ &\langle S_2|\tilde{L}^1|S_3\rangle - \langle S_3|\tilde{L}^{1+}|S_2\rangle, \end{aligned} \tag{23}$$

with the following notations used

$$\tilde{L}^0 = -p_\mu{}^2 + m^2 \tag{24}$$

$$\tilde{L}^1 = L^1 - ma_D \tag{25}$$

$$\tilde{L}_1^+ = L^{1+} - a_D^+ m, \tag{26}$$

The corresponding equations of motion are:

$$\begin{aligned}
\tilde{L}_0|S_1\rangle - \tilde{L}_1^+|S_3\rangle &= 0, \\
\tilde{L}_0|S_2\rangle - \tilde{L}_1|S_3\rangle &= 0, \\
|S_3\rangle - \tilde{L}_1|S_1\rangle + \tilde{L}_1^+|S_2\rangle &= 0.
\end{aligned} \tag{27}$$

While the gauge transformation law (22) in the component form looks as follows

$$\delta|S_1\rangle = \tilde{L}_1^+|\lambda\rangle, \tag{28}$$

$$\delta|S_2\rangle = \tilde{L}_1|\lambda\rangle, \tag{29}$$

$$\delta|S_3\rangle = \tilde{L}_0|\lambda\rangle. \tag{30}$$

Then using the gauge transformations one can show [10], that the fields $|S_2\rangle$ and $S_3\rangle$ as well as a_D^+ dependence in $|S_1\rangle$ can be gauged away. Finally one obtains conditions

$$(-p_\mu^2 + m^2)|S_1\rangle = L^1|S_1\rangle = 0 \tag{31}$$

as the result of the equations of motion.

Alternatively, one can first gauge away field $|S_2\rangle$ and a_D^+ dependence in fields $|S_1\rangle$ and $|S_3\rangle$, then express the field $|S_3\rangle$ using its own equation of motion in terms $|S_1\rangle$ and put back into the lagrangian (23) to obtain

$$L = -\langle S_1| - p_\mu^2 + (p_\mu a_\mu^+)(p_\nu a_\nu) + m^2|S_1\rangle \tag{32}$$

The lagrangian (32) (or (23)) describes all spins from zero to infinity with mass equal to m. Due to the luck of the tracelessness constraint, each spin n in the state $|S_1\rangle$ is followed also by spins $n-2$, $n-4$ ... on this level.

So, in this Section we demonstrated the method of BRST construction in the presence of two second class constraints $[L^1, L^{1+}] = -p_\mu^2 = L^0 - m^2$. The price for this was the introduction of additional creation and annihilation operators a_D^+ and a_D in the Fock space with corresponding modification of the representation for the basic operators L^1, L^{1+}. In the next Section the method will be generalized for more complicated systems of constraints.

3. IRREDUCIBLE MASSLESS CASE

3.1. BRST construction

The fields we are going to describe using the approach given in the previous Section correspond to the following Young tableaux

μ_1	μ_2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	μ_{n_1}
ν_1	ν_2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	ν_{n_2}		

$$(33)$$

and have the mass equal to zero i.e., $L^0 = -p_\mu^2$. Constraints describing these fields are those given in the first section, for $i, j = 1, 2$, namely $E^\alpha = (L^{ij}, T)$, $E^{-\alpha} = (L^{+ij}, T^+)$ and $\mathcal{L}^A = (L^0, L^i, L^{+i})$.

The operators $E^{\pm\alpha}$ and H^i form the algebra of group $SO(3,2)$, which can be written in the compact form as

$$\begin{aligned}
\left[H^i, E^\alpha\right] &= \alpha(i)E^\alpha, \\
\left[E^\alpha, E^{-\alpha}\right] &= \alpha^i H^i, \\
\left[E^\alpha, E^\beta\right] &= N^{\alpha\beta} E^{\alpha+\beta},
\end{aligned} \tag{34}$$

The corresponding nilpotent BRST charge for this subsystem of constraints can be constructed as follows [11].

First we construct the auxiliary representations of generators of the $SO(3,2)$ group using the Verma module after introduction of the additional creation and annihilation operators $\left[b_I, b_J^+\right] = \delta_{IJ}$, $I, J = 1, ..., 4$ The number of the oscillators is equal to the number of positive roots of the algebra $SO(3,2)$ and the vector $|\Phi\rangle$ depends also on the creation operators b_I^+.

Namely let us introduce the vector in the space of Verma module

$$\begin{aligned}
|n_1, n_2, n_3, n_4\rangle_V &= \\
(L^{11+})^{n_1}(L^{12+})^{n_2}&(L^{22+})^{n_3}(T^+)^{n_4}|0\rangle_V
\end{aligned} \tag{35}$$

where $n_i \in N$ and $E^\alpha|0\rangle_V = 0$. The corresponding vector in the Fock space generated by the creation and operators b_I^+ is

$$\begin{aligned}
|n_1, n_2, n_3, n_4\rangle &= \\
(b_1^+)^{n_1}(b_2^+)^{n_2}&(b_3^+)^{n_3}(b_4^+)^{n_4}|0\rangle.
\end{aligned} \tag{36}$$

Mapping the vectors in the Verma module onto the vectors in the Fock space one obtains the representations of the generators of $SO(3,2)$ (see [12]

for the general construction) algebra in terms of b_I, b_I^+ and constant parameters h^i, which characterize the highest weight representation $H_i|0\rangle_V = h_i|0\rangle_V$. However since the vectors in the Fock space form orthogonal basis and the corresponding vectors in Verma module do not, the correspondence between this two spaces is incomplete. In order to establish the complete correspondence one has to modify the scalar product in the Fock space under the condition

$$\langle \Phi_1|K|\Phi_2\rangle = {}_V\langle\Phi_1|\Phi_2\rangle_V. \tag{37}$$

with he Kernel operator K

$$K = Z^+Z, \tag{38}$$

$$Z = \sum_{n_i}(L^{11+})^{n_1}(L^{12+})^{n_2}(L^{22+})^{n_3}(T^+)^{n_4}|0\rangle_V \times$$

$$\langle 0|(b_1)^{n_1}(b_2)^{n_2}(b_3)^{n_3}(b_4)^{n_4}\frac{1}{\prod n_i!}. \tag{39}$$

Next define

$$\mathcal{H}^i = H^i + \tilde{H}^i_{aux.} + h^i, \tag{40}$$

$$\mathcal{E}^{\pm\alpha} = E^{\pm\alpha} + E^\alpha_{aux.}(h), \tag{41}$$

where we have explicitly extracted the dependence on parameters h_i in the auxiliary representations of Cartan generators. The corresponding nilpotent BRST charge with no H^i dependence has the form

$$\tilde{Q}_1 = \sum_{\alpha>0}\left(\eta_\alpha\mathcal{E}^{-\alpha} + \eta_{-\alpha}\mathcal{E}^\alpha\right) -$$

$$\frac{1}{2}\sum_{\alpha\beta}N^{\alpha\beta}\eta_{-\alpha}\eta_{-\beta}\mathcal{P}_{\alpha+\beta} \tag{42}$$

where, as a consequence of the procedure of dimensional reduction described in the previous Section the parameters h^i have to be substituted by the expressions

$$-\pi^i = -H^i - \tilde{H}^i_{aux} - \sum_{\beta>0}\beta(i)\left(\eta_\beta\mathcal{P}_{-\beta} - \eta_{-\beta}\mathcal{P}_\beta\right). \tag{43}$$

Indeed, after the construction of the standard nilpotent BRST charge Q_1 treating $\mathcal{E}^{\pm\alpha}$ and $\mathcal{H}^i$ as first class constraints, one considers the auxiliary phase space (x_i, p^i) with $p^i = h^i$. After that one makes the similarity transformation

$\tilde{Q}_1 = e^{i\pi^i x_i}Q_1 e^{-i\pi^i x_i}$ analogous to the one considered in the previous section. This transformation removes the dependence on the η_i ghost variables, which correspond to the Cartan generators $\mathcal{H}^i$. Then the $\mathcal{P}_i$ independent part of the transformed BRST charge is nilpotent and equals to (42).

The inclusion of the constraints $\mathcal{L}^\mathcal{A} \equiv (L^0, L^i, L^{i+})$ into the total BRST charge Q is trivial, namely

$$\tilde{Q} = \tilde{Q}_1 + \tilde{Q}_2 \tag{44}$$

where

$$\tilde{Q}_2 = \eta_0 L^0 + \eta_i L^{i+} + \eta_i^+ L^i - \eta_i^+\eta_i\mathcal{P}_I +$$

$$\sum_{\alpha>0,A,B}(\eta_A\eta_\alpha^+\mathcal{P}_\mathcal{B}\mathcal{C}^\mathcal{B}_{-\alpha,\mathcal{A}} + \eta_\mathcal{A}\eta_\alpha\mathcal{P}_\mathcal{B}\mathcal{C}^\mathcal{B}_{\alpha,\mathcal{A}}) \tag{45}$$

in self explanatory notations. This completes the procedure of constructing nilpotent BRST charge for our system.

The BRST invariant lagrangian can be written as

$$-L = \int d\eta_0\langle\chi|K\tilde{Q}|\chi\rangle, \tag{46}$$

being invariant under the gauge transformations

$$\delta|\chi\rangle = \tilde{Q}|\Lambda\rangle \tag{47}$$

Following the lines of the previous section, after the integration over the ghost variables and elimination of auxiliary fields using the equations of motion and the BRST gauge transformations one arrives to the final expression for the lagrangian [13] describing all massless irreducible representations of the Poincare group with the corresponding Young tableaux having two rows

$-L =$
$\langle S_1|L^0 - L^{+1}L^1 - L^{+2}L^2 - L^{+1}L^{+1}L^{11} -$
$L^{+1}L^{+2}L^{12} - L^{+2}L^{+2}L^{22} - 2L^{+11}L^0L^{11} -$
$L^{+11}L^1L^1 - L^{+12}L^0L^{12} - L^{+12}L^1L^2 -$
$2L^{+22}L^0L^{22} - L^{+22}L^2L^2 -$
$L^{+1}L^{+11}L^1L^{11} - L^{+1}L^{+12}L^2L^{11} +$
$L^{+1}L^{+22}L^1L^{22} - L^{+1}L^{+22}L^2L^{12} -$
$L^{+2}L^{+11}L^1L^{12} + L^{+2}L^{+11}L^2L^{11} -$
$L^{+2}L^{+12}L^1L^{22} - L^{+2}L^{+22}L^2L^{22} +$

$$L^{+1}L^{+1}L^{+22}L^{11}L^{22} - L^{+1}L^{+2}L^{+12}L^{11}L^{22} +$$
$$L^{+2}L^{+2}L^{+11}L^{11}L^{22} + 3L^{+11}L^{+22}L^0L^{11}L^{22} +$$
$$L^{+11}L^{+22}L^1L^1L^{22} - L^{+11}L^{+22}L^1L^2L^{12} +$$
$$L^{+11}L^{+22}L^2L^2L^{11} +$$
$$L^{+1}L^{+11}L^{+22}L^1L^{11}L^{22} +$$
$$L^{+2}L^{+11}L^{+22}L^2L^{11}L^{22}|S_1\rangle \qquad (48)$$

where the field $|S_1\rangle$ is constrained as

$$T|S_1\rangle = 0 \qquad (49)$$

$$L^{\{ij}L^{kl\}}|S_1\rangle = 0 \qquad (50)$$

In particular, the equation (49) leads also to the following property of the basic field

$$\Phi_{\mu_1\mu_2\cdots\mu_n,\nu_1\nu_2\cdots\nu_n}(x) =$$
$$(-1)^n\Phi_{\nu_1\nu_2\cdots\nu_n,\mu_1\mu_2\cdots\mu_n}(x), \qquad (51)$$

when the numbers of indices μ and ν coincide.

The lagrangian (48) is invariant under the transformations

$$\delta|S_1\rangle = (L^{1+} + L^{2+}T)|\lambda\rangle \qquad (52)$$

with the parameter of gauge transformations $|\lambda\rangle$ constrained as follows

$$L^{ij}|\lambda\rangle = T^2|\lambda\rangle = 0 \qquad (53)$$

obviously the constraints on the basic field (49)-(50) are also invariant under the gauge transformations (52).

Let us note, that neglecting the a_μ^{2+} dependence in the field $|S_1\rangle$ one obtains the lagrangian given in [6] for irreducible massless higher spin fields.

3.2. Examples

Let us construct the explicit form of the lagrangians for some simple Young tableaux which correspond to lower orders in the expansion of the field $|S_0\rangle$.

- ☐ : $|S_1\rangle = \Phi_{\mu,\nu}(x)a_\mu^{1+}a_\nu^{2+}|0\rangle$

The lagrangian (48) for this antisymmetric field in terms of the field strength $F_{\mu\nu\rho} = \partial_\mu\Phi_{\nu,\rho} + \partial_\nu\Phi_{\rho,\mu} + \partial_\rho\Phi_{\mu,\nu}$ has the standard form

$$L = \frac{1}{3}F_{\mu\nu\rho}^2 \qquad (54)$$

and is invariant under the gauge transformations

$$\delta\Phi_{\mu,\nu}(x) = \partial_\mu\lambda_\nu(x) - \partial_\nu\lambda_\mu(x). \qquad (55)$$

- ☐☐ (vertical) : $|S_1\rangle = \Phi_{\mu\nu,\rho}(x)a_\mu^{1+}a_\nu^{1+}a_\rho^{2+}|0\rangle$

The symmetry with respect to the first two indices $\Phi_{\mu\nu,\rho} = \Phi_{\nu\mu,\rho}$ which is guaranteed by the construction and the condition (49) lead to the following property of the field $\Phi_{\mu\nu,\rho}$:

$$\Phi_{\mu\nu,\rho} + \Phi_{\rho\nu,\mu} + \Phi_{\mu\rho,\nu} = 0. \qquad (56)$$

The lagrangian (48) for the third rank tensor field $\Phi_{\mu\nu,\rho}$ can be written in the form

$$L = 2\Phi_{\mu\nu,\rho}\partial_\sigma^2\Phi_{\mu\nu,\rho} - 3\Phi_{\mu\mu,\rho}\partial_\sigma^2\Phi_{\nu\nu,\rho} -$$
$$4\Phi_{\mu\nu,\rho}\partial_\mu\partial_\sigma\Phi_{\sigma\nu,\rho} - 2\Phi_{\mu\nu,\rho}\partial_\rho\partial_\sigma\Phi_{\mu\nu,\sigma} +$$
$$6\Phi_{\mu\nu,\rho}\partial_\mu\partial_\nu\Phi_{\sigma\sigma,\rho} + 3\Phi_{\mu\mu,\rho}\partial_\rho\partial_\sigma\Phi_{\nu\nu,\sigma}, \qquad (57)$$

and is invariant under the gauge transformations

$$\delta\Phi_{\mu\nu,\rho}(x) = \partial_\mu\lambda_{\nu,\rho}(x) + \partial_\nu\lambda_{\mu,\rho}(x) -$$
$$\partial_\rho\lambda_{\mu,\nu}(x) - \partial_\rho\lambda_{\nu,\mu}(x), \qquad (58)$$

the gauge transformation parameter being traceless $\lambda_{\mu,\mu} = 0$.

- ☐☐ (2x2) : $|S_1\rangle = \Phi_{\mu\nu,\rho\sigma}(x)a_\mu^{1+}a_\nu^{1+}a_\rho^{2+}a_\sigma^{2+}|0\rangle$

The field $\Phi_{\mu\nu,\rho\sigma}$ can be related to the field $C_{\mu\nu,\rho\sigma}$ which has symmetries of the Weyl tensor

$$C_{\mu\nu,\rho\sigma} = -C_{\nu\mu,\rho\sigma} = -C_{\mu\nu,\sigma\rho}, \qquad (59)$$

$$C_{\mu\nu,\rho\sigma} = C_{\rho\sigma,\mu\nu}, \qquad (60)$$

in terms of relations

$$\Phi_{\mu\nu,\rho\sigma} = \frac{1}{4}(C_{\mu\rho,\nu\sigma} + C_{\mu\sigma,\nu\rho}) \qquad (61)$$

$$C_{\mu\rho,\nu\sigma} = \frac{4}{3}(\Phi_{\mu\nu,\rho\sigma} - \Phi_{\rho\nu,\mu\sigma}). \qquad (62)$$

Strictly speaking $C_{\mu\nu,\rho\sigma}$ is not a Weyl tensor because its traces do not vanish. However, one can obtain from (48) the following lagrangian

$$L = -\frac{1}{2}C_{\mu\rho,\nu\tau}\partial_\sigma^2C_{\mu\rho,\nu\tau} - \frac{1}{2}C_{\mu\tau,\nu\rho}\partial_\sigma^2C_{\mu\rho,\nu\tau} +$$
$$3C_{\mu\rho,\nu\sigma}\partial_\sigma^2C_{\mu\tau,\nu\tau} - \frac{3}{4}C_{\mu\nu,\mu\nu}\partial_\sigma^2C_{\rho\tau,\rho\tau} +$$
$$2C_{\mu\rho,\nu\sigma}\partial_\mu\partial_\tau C_{\tau\rho,\nu\sigma} + 2C_{\mu\rho,\nu\sigma}\partial_\mu\partial_\tau C_{\tau\sigma,\nu\rho} -$$
$$6C_{\mu\rho,\nu\sigma}\partial_\mu\partial_\nu C_{\tau\rho,\tau\sigma} - 6C_{\mu\rho,\nu\sigma}\partial_\mu\partial_\sigma C_{\sigma\tau,\nu\tau} +$$
$$3C_{\mu\rho,\nu\sigma}\partial_\mu\partial_\nu C_{\sigma\tau,\sigma\tau} \qquad (63)$$

invariant under the gauge transformations

$$\delta C_{\mu\nu,ab}(x) = \partial_\mu \lambda_{\nu,ab}(x) - 2\partial_a \lambda_{\mu,\nu b}(x) - (\mu \leftrightarrow a, \nu \leftrightarrow b) \tag{64}$$

$$\lambda_{\mu,\mu\nu} = \lambda_{\mu,\nu\nu} = 0, \quad \lambda_{\mu,\nu\rho} = \lambda_{\mu,\rho\nu} \tag{65}$$

$$\lambda_{\mu,\nu\rho} + \lambda_{\nu,\rho\mu} + \lambda_{\rho,\mu\nu} = 0 \tag{66}$$

where the symmetrization over couples of Greek and Latin indices is assumed. It can be shown that the vanishing of all traces of this tensor on mass shell follows from this lagrangian . Therefore one can conclude that the lagrangian (63) consistently describes the free field theory of the Weyl tensor.

4. DISCUSSION

The approach we have described can be straightforwardly generalized to the particles which belong to an arbitrary representations of the Poincare group as well. Another challenging problem is construction of the theory of interacting higher spin fields. It is known that the particles with the higher spins can propagate through the background having the constant curvature, in particular through the AdS space (see [14] and the references therein), as well as interact with the constant electromagnetic field or symmetrical Einstein spaces [15]. The utilization of the technique of the Supersymmetric Quantum Mechanics leads also to the description of the particle with spin 2 on the background of the constant curvature [16], and on the background being real "Kahler – like" manifold [17]. In the highlight of these developments it seems interesting to generalize this technique also for the description of the interaction of higher spin fields with some gravitational background. The inclusion of gravitational background will obviously lead to the modification of the system of constraints presented in the BRST charge. The problem of constructing of the nilpotent BRST charge for these kinds of physical systems can in turn reveal an allowed types of gravitational backgrounds where the higher spin fields can propagate consistently.

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